Discretisation in Parabolic Heat Conduction using Transmission Line Matrix Modelling Method

Sheikh M Firoz
Department of Engineering

Aims of the Research

- The goal of this research was to investigate TLM (Transmission Line Matrix) parameterisations for parabolic heat conduction modelling.
- To investigate a method for optimum selection of elemental size (Δl) and time step (Δt) for the parabolic modelling using State-Space Analysis.
- To assess the feasibility of these methods via simulations.

Parabolic Heat Conduction

- Parabolic heat conduction, also known as the parabolic heat equation or the transient heat conduction equation.

- A mathematical model used to describe the transient temperature distribution in a solid medium over time.

- It is derived from Fourier's law of heat conduction and is based on the assumption that the thermal conductivity of the medium is constant.

The parabolic heat conduction equation is a partial differential equation (PDE). It can be expressed as:

$\rho c\, \partial T/\partial t = k\, \nabla^2 T$ where ρ is the density of the material
c is the specific heat capacity
T is the Temperature in terms of space & Time
t is the time
k is the thermal conductivity
$\partial T/\partial t$ is the rate of change of temperature with respect to time
$\nabla^2 T$ is the Laplacian operator of temperature with respect to the spatial coordinates x, y, z.

Transmission Line Matrix (TLM) Modelling Method

- A General Numerical Technique in solving transient field propagation.
- In TLM models, physical meaning is preserved.
- Numerical routines based on TLM are explicit, one-step and unconditionally stable.
- In thermal diffusion modelling- the volume is divided into elements each with nodes at which the temperature is calculated, and the modelling period is divided into iteration time steps.
- In a TLM network, current and electrical potential are analogous to heat flow and temperature in the physical world.
- In 1977, P. B. Johns described in his research paper that the TLM method could be developed for the solution of heat diffusion as well as in the electromagnetic field.

Investigation Stages

- TLM to TLM State-Space
- Parabolic Diffusion Equation
- Parabolic TLM Parameterisation
- Simulation Setup
- Selection of space step and time step

The Simulation Process

Geometry

- Representation of Experimental Situation including material parameters

Fourier Analysis to compute wave numbers

- Spatial resolution made to find the optimum spatial wave number to guide wave number to simulate

Discretisation Curves to estimate space step (Δl) and time step (Δt)

- Representing the continuous coefficient close to the curve corresponding to the impulse invariant response.

Parameterisation of TLM Models

- Parabolic parameterisations

TLM Simulations

- Parabolic Simulations based on materials properties

TLM to TLM State-Space

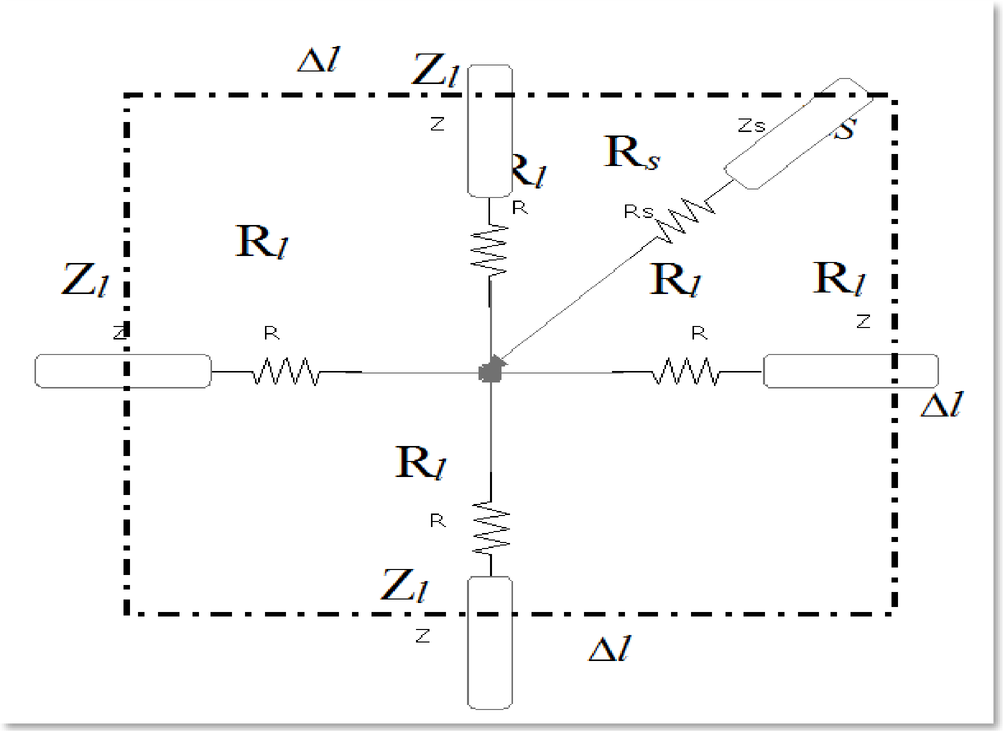


Figure-1: A TLM network with stub loaded that all the R_ls and Z_ls in the node are the same value for square element.

- Two types of transmission line- Link (R_l and Z_l) and Stub (R_S and Z_S)transmission Line
- The Stub line is particularly useful in dealing with non-uniform or irregular mesh modelling problems
- Stubs also have a ‘smoothing’ effect on the algorithm output and sometimes are used purely for this purpose.

TLM to TLM State-Space

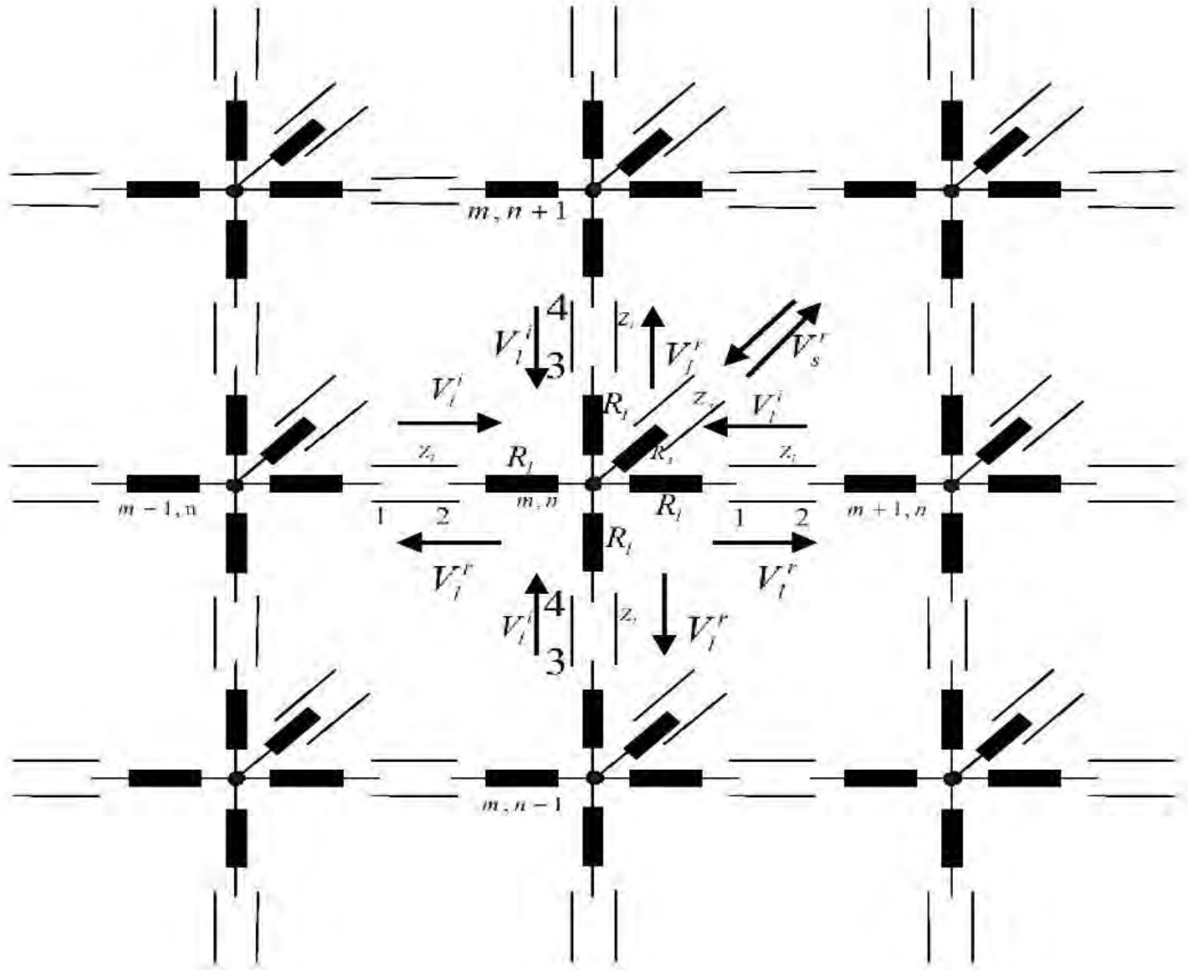


Figure-2: TLM Pulse Scattering in a 2-D node.

TLM to TLM State-Space

TLM nodal potential equation $${}_k\varphi = \left(\sum_{l=1}^{l=2\eta} \frac{2\,{}_kV_l^i}{R_l + Z_l} + \frac{2\,{}_kV_s^i}{R_s + Z_s} + c_r \right) \frac{1}{Y} \quad (2.18)$$

where $$Y = \left(\sum_{l=1}^{l=2\eta} \frac{1}{R_l + Z_l} \right) + \frac{1}{R_s + Z_s}$$

and C_r is a nodal current source input

Thévenin equivalent circuit of a node is: $${}_kV_l^r = {}_k\phi \frac{Z_l}{R_l + Z_l} + {}_kV_l^i \frac{R_l - Z_l}{R_l + Z_l} \quad (2.12)$$

For the stub transmission line is: $${}_kV_s^r = {}_k\phi \frac{Z_s}{R_s + Z_s} + {}_kV_s^i \frac{R_s - Z_s}{R_s + Z_s} \quad (2.13)$$

$$\begin{aligned}
{}_{k+1}V_{m,n,s}^i &= {}_kV_{m,n,s}^r \\
{}_{k+1}V_{m-1,n,1}^i &= {}_kV_{m,n,2}^r \\
{}_{k+1}V_{m+1,n,2}^i &= {}_kV_{m,n,1}^r \qquad (2.9) \\
{}_{k+1}V_{m,n-1,3}^i &= {}_kV_{m,n,4}^r \\
{}_{k+1}V_{m,n+1,4}^i &= {}_kV_{m,n,3}^r
\end{aligned}$$

TLM to TLM State-Space

TLM discrete state-space representation:

$$\underline{QI}\,{}_kV^i = \underline{Sc}\,{}_kV^i + \underline{B}u_{(k)} \quad (2.21) \quad \text{and} \quad {}_k\varphi = \underline{C}\,{}_kV^i + Du_{(k)} \quad (2.22)$$

The dynamic relationship between input and output:

$${}_kV^i = (\underline{QI} - \underline{Sc})^{-1}\underline{B}u_{(k)} \quad (2.23) \qquad {}_k\varphi = \underline{C}(\underline{QI} - \underline{Sc})^{-1}\underline{B}u_{(k)} + Du_{(k)} \quad (2.24)$$

The transfer function relating φ and u can then be re-expressed as:

$${}_k\varphi = (\underline{C}(\underline{QI} - \underline{Sc})^{-1}\underline{B} + D)u_k \quad (2.25)$$

The overall transfer function for a general 2-D TLM node is:

$${}_k\phi = \left(\frac{Driving\ Force}{\det\left|\underline{QI} - \underline{Sc}\right|}\right)u_k \qquad \text{or} \qquad \det\left|\underline{QI} - \underline{Sc}\right|\ {}_k\phi = Driving\ Force\ u_k$$

The driving force or forcing term is represented by the numerator of the right-hand side of equation (2.25):

$$numerator\ \ \underline{C}adj\ \ \underline{QI} - \underline{Sc}\ \ \underline{B} + D \quad (2.28)$$

$$Driving\_force = \frac{QR_l + QZ_l + Z_l - R_l \quad QR_l + QZ_l + R_l - Z_l}{2QZ_l} \quad (2.29)$$

TLM to TLM State-Space

$$\underline{B} = \begin{bmatrix} \dfrac{Z_s}{(R_s + Z_s)\left(\dfrac{1}{R_s + Z_s} + \dfrac{4}{R_l + Z_l}\right)} \\ \dfrac{Z_l}{(R_l + Z_l)\left(\dfrac{1}{R_s + Z_s} + \dfrac{4}{R_l + Z_l}\right)} \\ \dfrac{Z_l}{(R_l + Z_l)\left(\dfrac{1}{R_s + Z_s} + \dfrac{4}{R_l + Z_l}\right)} \\ \dfrac{Z_l}{(R_l + Z_l)\left(\dfrac{1}{R_s + Z_s} + \dfrac{4}{R_l + Z_l}\right)} \\ \dfrac{Z_l}{(R_l + Z_l)\left(\dfrac{1}{R_s + Z_s} + \dfrac{4}{R_l + Z_l}\right)} \end{bmatrix}$$

$$\underline{C} = \begin{bmatrix} \dfrac{2}{(R_s + Z_s)\left(\dfrac{1}{R_s + Z_s} + \dfrac{4}{R_l + Z_l}\right)} \\ \dfrac{2}{(R_l + Z_l)\left(\dfrac{1}{R_s + Z_s} + \dfrac{4}{R_l + Z_l}\right)} \\ \dfrac{2}{(R_l + Z_l)\left(\dfrac{1}{R_s + Z_s} + \dfrac{4}{R_l + Z_l}\right)} \\ \dfrac{2}{(R_l + Z_l)\left(\dfrac{1}{R_s + Z_s} + \dfrac{4}{R_l + Z_l}\right)} \\ \dfrac{2}{(R_l + Z_l)\left(\dfrac{1}{R_s + Z_s} + \dfrac{4}{R_l + Z_l}\right)} \end{bmatrix}$$

$$D = \frac{1}{\dfrac{1}{R_s + Z_s} + \dfrac{4}{R_l + Z_l}}$$

11

TLM to TLM State-Space

$\underline{Sc}$ = [redacted]

TLM to TLM State-Space

The driving force or forcing term is represented by the numerator of the right-hand side of equation and left-hand side of the equation is represented by denominator which correspond to the characteristic equation represented by the TLM node.

$$\left| \frac{Q-1 \quad R_l+QR_l-Z_l+QZ_l \quad QR_l+QZ_l+4QR_s+4QZ_s-R_l-4R_s+4Z_s+Z_l}{2Z_lQ \;\; QR_s+QZ_s+Z_s-R_s} - \;\; I-2+I^{-1} \;\; - \;\; J-2+J^{-1} \;\; \right|_k \phi$$

$$=\left(\frac{QR_l+QZ_l+Z_l-R_l \quad QR_l+QZ_l+R_l-Z_l}{2QZ_l} \right) u_k$$

Parabolic Diffusion Equation

Two-dimensional parabolic thermal diffusion equation is: $$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{S\rho}{K_T}\frac{\partial T}{\partial t} \qquad (2.2)$$

$\rho c\, \partial T/\partial t = k\, \nabla^2 T$

A discrete description of the TLM parabolic heat conduction equation (2.2) for a 2-dimensional model is:

$$\frac{{}_kT_{m+1,n} - 2\,{}_kT_{m,n} + {}_kT_{m-1,n}}{\Delta l^{\,2}} + \frac{{}_kT_{m,n-1} - 2\,{}_kT_{m,n} + {}_kT_{m+1,n}}{\Delta l^{\,2}} = \frac{S\rho}{k_T}\,\frac{{}_{k+1}T_{m,n} - {}_kT_{m,n}}{\Delta t} \qquad (2.30)$$

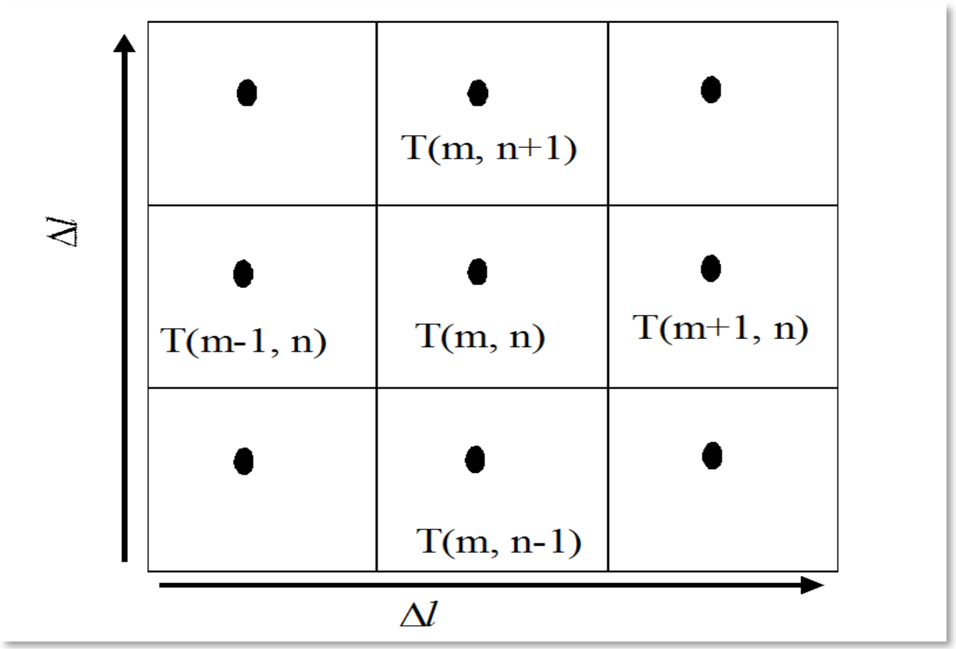


Figure-3: Spatial Relationship of Temperature represented to discretisation of parabolic heat propagation.

Parabolic TLM Parameterisation

$$\frac{Q-1\quad R_l+QR_l-Z_l+QZ_l\quad QR_l+QZ_l+4QR_s+4QZ_s-R_l-4R_s+4Z_s+Z_l}{2Z_lQ\quad QR_s+QZ_s+Z_s-R_s} - \quad I-2+I^{-1} \quad - \quad J-2+J^{-1} \quad = 0 \qquad (2.44)$$

Parameterisation of nodal structure: $R_l = Z_l = \frac{1}{w} and\, R_s = Z_s = \frac{1}{Y_s \times w} \quad (2.45)$

Relating the derivation of the discrete state-space analogy to the TLM equation (2.45) and the denominator of equation (2.44), the characteristic equation becomes:

$$\frac{Q \quad 1 .\left[\frac{1}{w} \quad Q.\ \frac{1}{w} \quad \frac{1}{w} \quad Q.\ \frac{1}{w}\right] Q.\ \frac{1}{w} \quad Q.\ \frac{1}{w} \quad 4.Q.\left(\frac{1}{Y_s \times w}\right) \quad 4.Q.\left(\frac{1}{Y_s \times w}\right) \quad \frac{1}{w} \quad 4.\left(\frac{1}{Y_s \times w}\right) \quad 4.\left(\frac{1}{Y_s \times w}\right) \quad \frac{1}{w}}{2.\left(\frac{1}{w}\right).Q.\left[Q.\left(\frac{1}{Y_s \times w}\right)+Q.\left(\frac{1}{Y_s \times w}\right)+\left(\frac{1}{Y_s \times w}\right)-\left(\frac{1}{Y_s \times w}\right)\right]} - \left| I-2+I^{-1} \right| - \left| J-2+J^{-1} \right| = 0$$

.........(2.46)

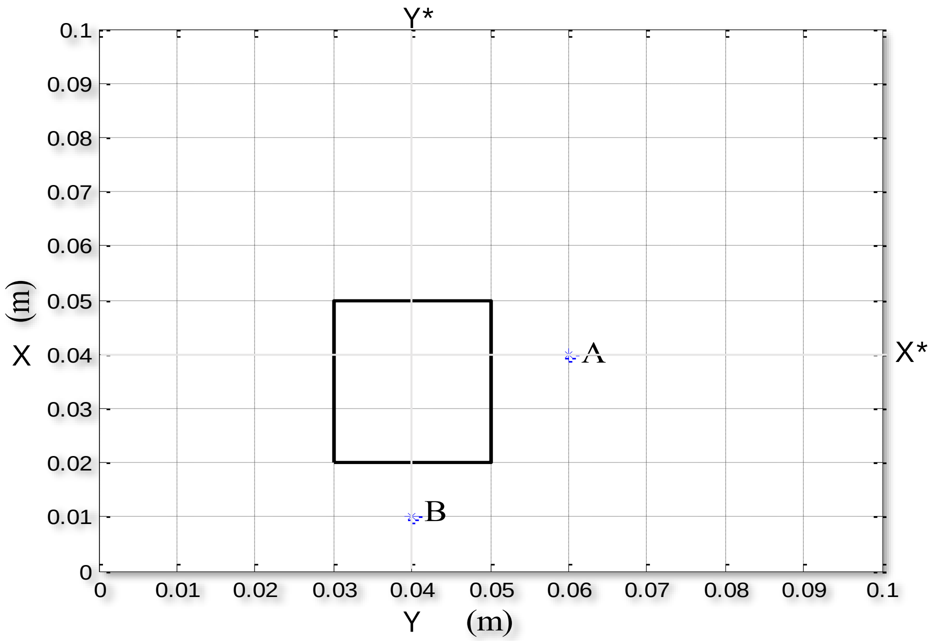
$$w = \frac{1}{(\Delta l)^2} \qquad (2.47)$$

Arrangement of equations (2.46 and 2.47), the diffusion process (Q-1) representation becomes:

$$\frac{\left(I-2+I^{-1}\right)}{(\Delta l)^2} - \frac{\left(J-2+J^{-1}\right)}{(\Delta l)^2} = \frac{\Delta t}{(\Delta l)^2}.\left(Y_s+4\right).\frac{\left(Q-1\right)}{\Delta t} \qquad (2.48)$$

$$\frac{\Delta t}{(\Delta l)^2}.\left(Y_s+4\right) = \frac{S\rho}{K_T} \qquad Y_s = \frac{S\rho(\Delta l)^2 - 4K_T(\Delta t)}{K_T(\Delta t)}$$

Simulation Setup



Thermal Properties of Alluminium

$$K_T = 2.5 W m^{-1} K^{-1}$$

$$s = 500 J K g^{-1}$$

$$\rho = 1000 K g m^{-3}$$

Figure 5: A typical layout of the physical cross-section arrangement for the discretisation of numerical simulations.

- The slab of material insulated on all sides and having a cross-section which used to compute wave numbers (α^2) needed as model parameters.
- Initial condition of the temperature is 0^0C
- The square region instantaneously acquires a temperature of 200^0C at the time of 0s

Selection of Space Step and Time Step

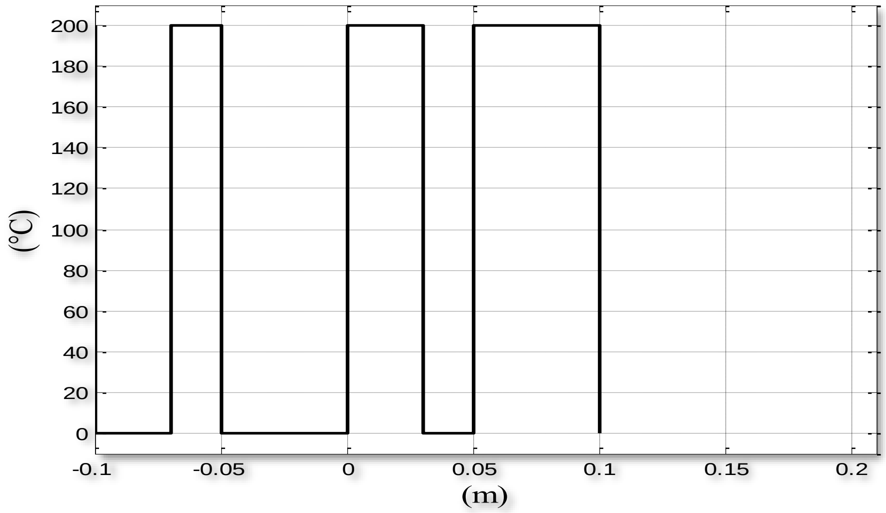


Figure 6: The physical cross-section to evaluate Fourier coefficient in the x directions.

- The Fourier series assumes as a periodic structure, so an inverted wave form is added in order to ensure that the spatial boundary of the problem is identified.

For the x-direction from X to X* cross-section Fourier series is

$$f(x) \approx \sum_{n=-\infty}^{\infty} c_n e^{jn\pi x/l} \quad where \quad c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-jn\pi x/l}\, dx$$

$$c_n = \frac{1}{0.2}\int_{-0.1}^{0.1} f(x).e^{\left(\frac{-jn\pi x}{0.1}\right)}dx = \frac{200}{0.2}\left[\frac{e^{\left(\frac{-jn\pi x}{0.1}\right)}}{\frac{-jn\pi}{0.1}}\right]_{-0.07}^{-0.05} + \frac{200}{0.2}\left[\frac{e^{\left(\frac{-jn\pi x}{0.1}\right)}}{\frac{-jn\pi}{0.1}}\right]_{0}^{0.03} + \frac{200}{0.2}\left[\frac{e^{\left(\frac{-jn\pi x}{0.1}\right)}}{\frac{-jn\pi}{0.1}}\right]_{0.05}^{0.1} \qquad (2.51)$$

so that

$$c_n = 20e^{jn0.6\pi} sinc(n0.1\pi) + 30e^{-jn0.15\pi} sinc(n0.15\pi) + 50e^{jn0.75\pi} sinc(n0.25\pi)$$

Selection of Space Step and Time Step

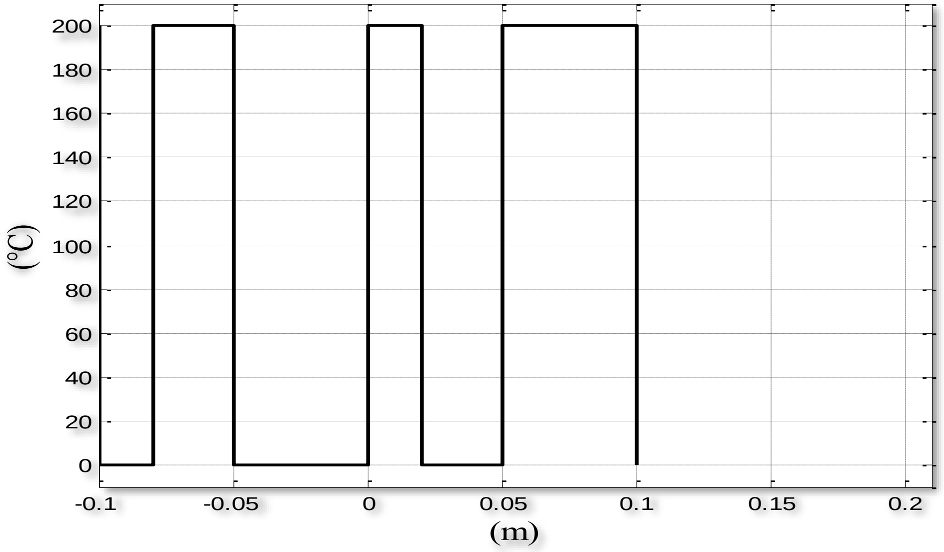


Figure 7: The physical cross-section to evaluate Fourier coefficient in the y directions.

where $\alpha = \frac{n\pi}{l} = \frac{10\pi}{0.1} = 100\pi$

For the y-direction from cross section Y to Y*, Fourier series is

$$c_n = \frac{1}{0.2}\int_{-0.1}^{0.1} f\left(y\right) e^{\left(\frac{-jn\pi y}{0.1}\right)} dy = \frac{200}{0.2}\left[\frac{e^{\frac{-jn\pi y}{0.1}}}{\frac{-jn\pi}{0.1}}\right]_{-0.08}^{-0.05} + \frac{200}{0.2}\left[\frac{e^{\frac{-jn\pi y}{0.1}}}{\frac{-jn\pi}{0.1}}\right]_{0}^{0.02} + \frac{200}{0.2}\left[\frac{e^{\frac{-jn\pi y}{0.1}}}{\frac{-jn\pi}{0.1}}\right]_{0.05}^{0.1} \quad (2.52)$$

$so \quad that$

$$c_n = 30e^{jn0.65\pi} sinc\left(n\pi 0.15\right) + 20e^{jn0.1\pi} sinc\left(n\pi 0.1\right) + 50e^{-jn0.75\pi} sinc\left(n\pi 0.25\right)$$

Parabolic Diffusion Equation

Assuming the general solution to the diffusion equation is of the form: $T(x, y, t) = X(x)Y(y)P(t)$

For the time derivative of the diffusion equation : $\dot{P} + \sigma^2 \frac{K_T}{S\rho} P = 0 \qquad (2.33)$

$$\ddot{X} + \alpha^2 X = 0 \text{ and } \ddot{Y} + \alpha^2 Y = 0 \qquad \text{where} \qquad \sigma^2 = \alpha^2 + \alpha^2$$

For the x component the impulse invariant solution is:

$$\left[\frac{I - 2 - I^{-1}}{\Delta x^2}\right] + \left[\frac{2 - e^{-j\alpha\Delta x} - e^{j\alpha\Delta x}}{\Delta x^2}\right] X = \left[\frac{\left[\frac{e^{-j\alpha\Delta x} - 1}{-j\alpha}\right]\left[\frac{e^{j\alpha\Delta x} - 1}{j\alpha}\right]}{\Delta x^2}\right] F_X \qquad (2.40)$$

The impulse invariant solution for the time component is:

$$\left[\frac{Q-1}{\Delta t} + \frac{1 - e^{-\sigma^2 K_T \Delta t}}{\Delta t}\right] P = \frac{\left[\frac{e^{\frac{-\sigma^2 K_T \Delta t}{S\rho}} - 1}{\frac{-\sigma^2 K_T}{S\rho}}\right]}{\Delta t} F_P \qquad (2.42)$$

Parabolic Discretisation

The choice of discretisation has been made by comparing discrete and continuous solution coefficient using separable variables equations to determine Δl and Δt are as:

for *x*-directions

$$\alpha^2 \approx \frac{2 - e^{-j\alpha\Delta l} - e^{j\alpha\Delta l}}{(\Delta x)^2} \qquad (2.53)$$

for *y*-directions

$$\alpha^2 \approx \frac{\left(2 - e^{-j\alpha\Delta l} - e^{j\alpha\Delta l}\right)}{(\Delta y)^2} \qquad (2.54)$$

$$\sigma^2 = \alpha^2 + \alpha^2 = \frac{\left(2 - e^{-j\alpha\Delta l} - e^{j\alpha\Delta l}\right)}{(\Delta l)^2} + \frac{\left(2 - e^{-j\alpha\Delta l} - e^{j\alpha\Delta l}\right)}{(\Delta l)^2} \qquad (2.56)$$

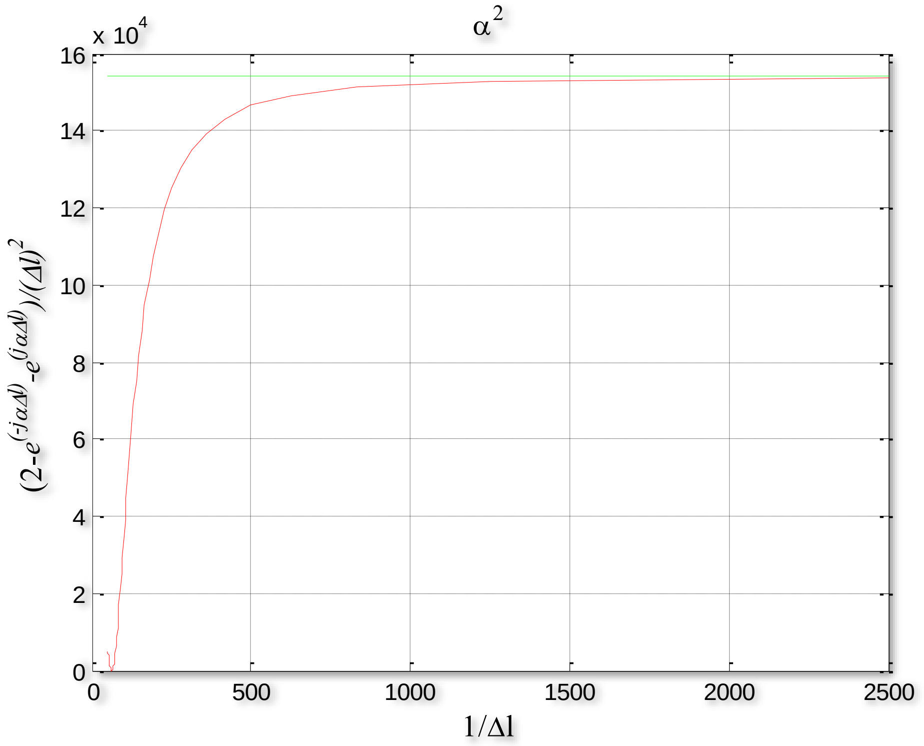


Figure 8: Plot of discretised coefficient for α^2 against $1/\Delta l$

Parabolic Discretisation

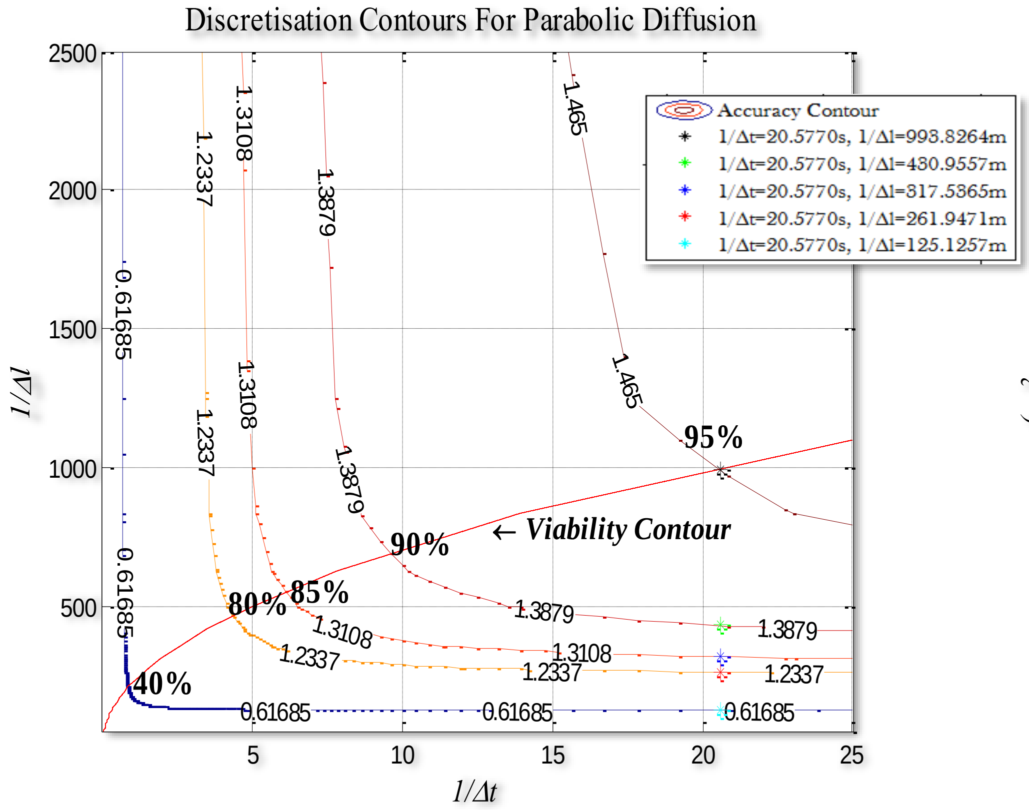


Figure 9:Discretisation contours from which to choose percentage ‘accuracy’.

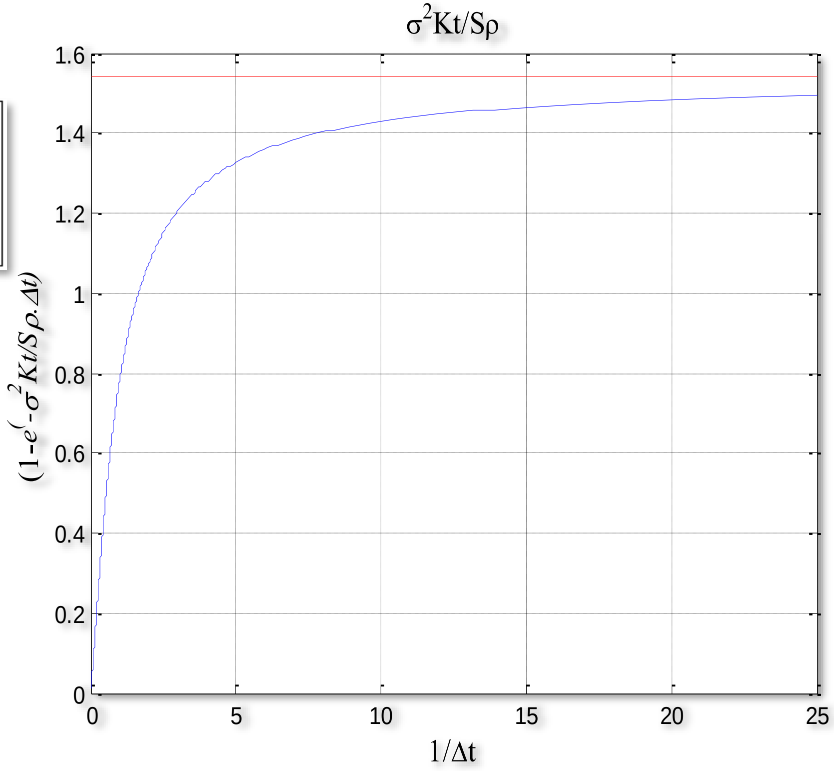


Figure 10: Plot of coefficient for $\frac{\sigma^2 K_T}{S\rho}$ *against* $1/\Delta t$

And for the time coefficient: $$\sigma^2 \frac{K_T}{S\rho} \approx \frac{1-e^{\frac{-\sigma^2 K_T \Delta t}{S\rho}}}{\Delta t}$$

Parabolic Simulation Results

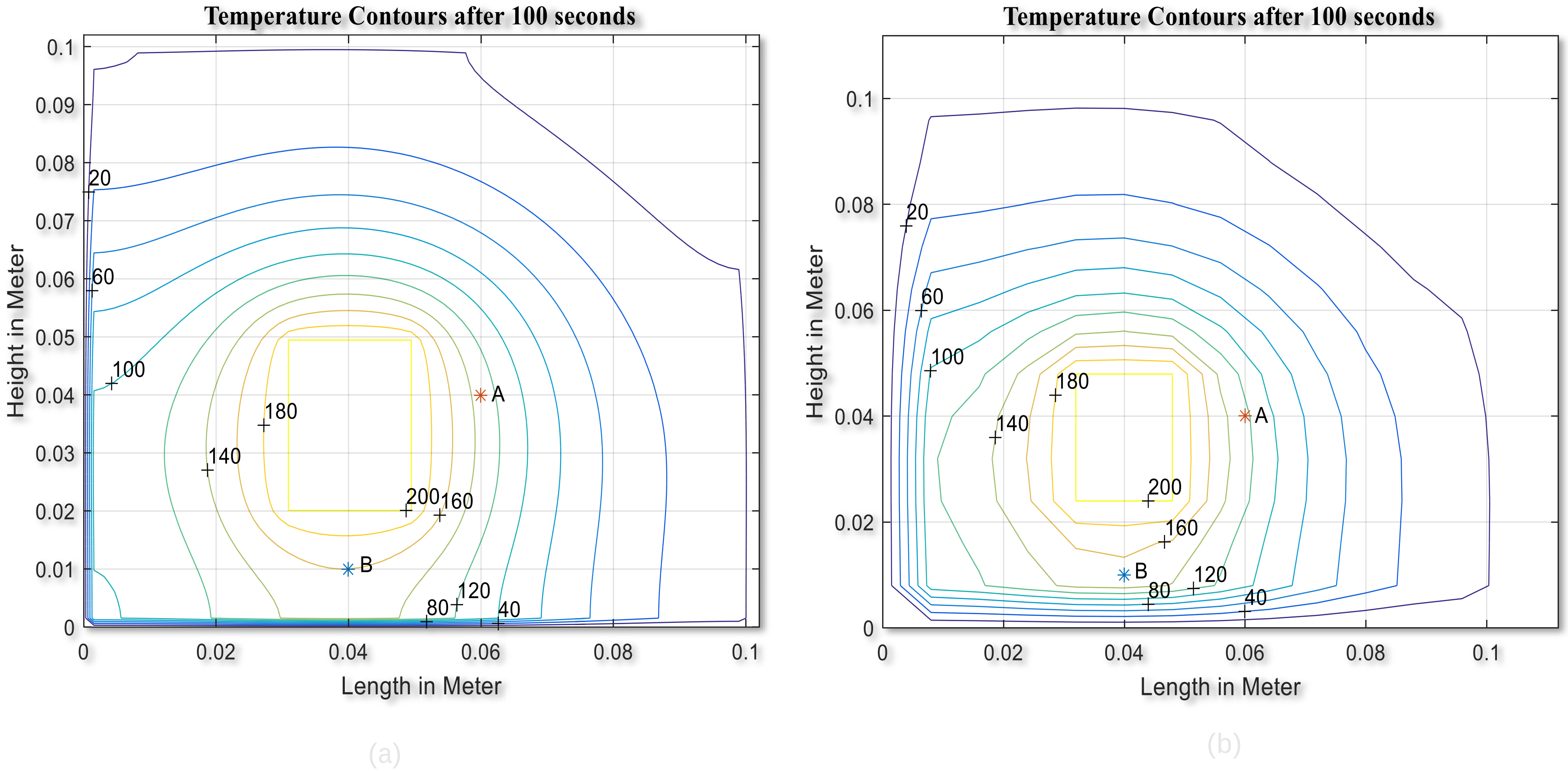
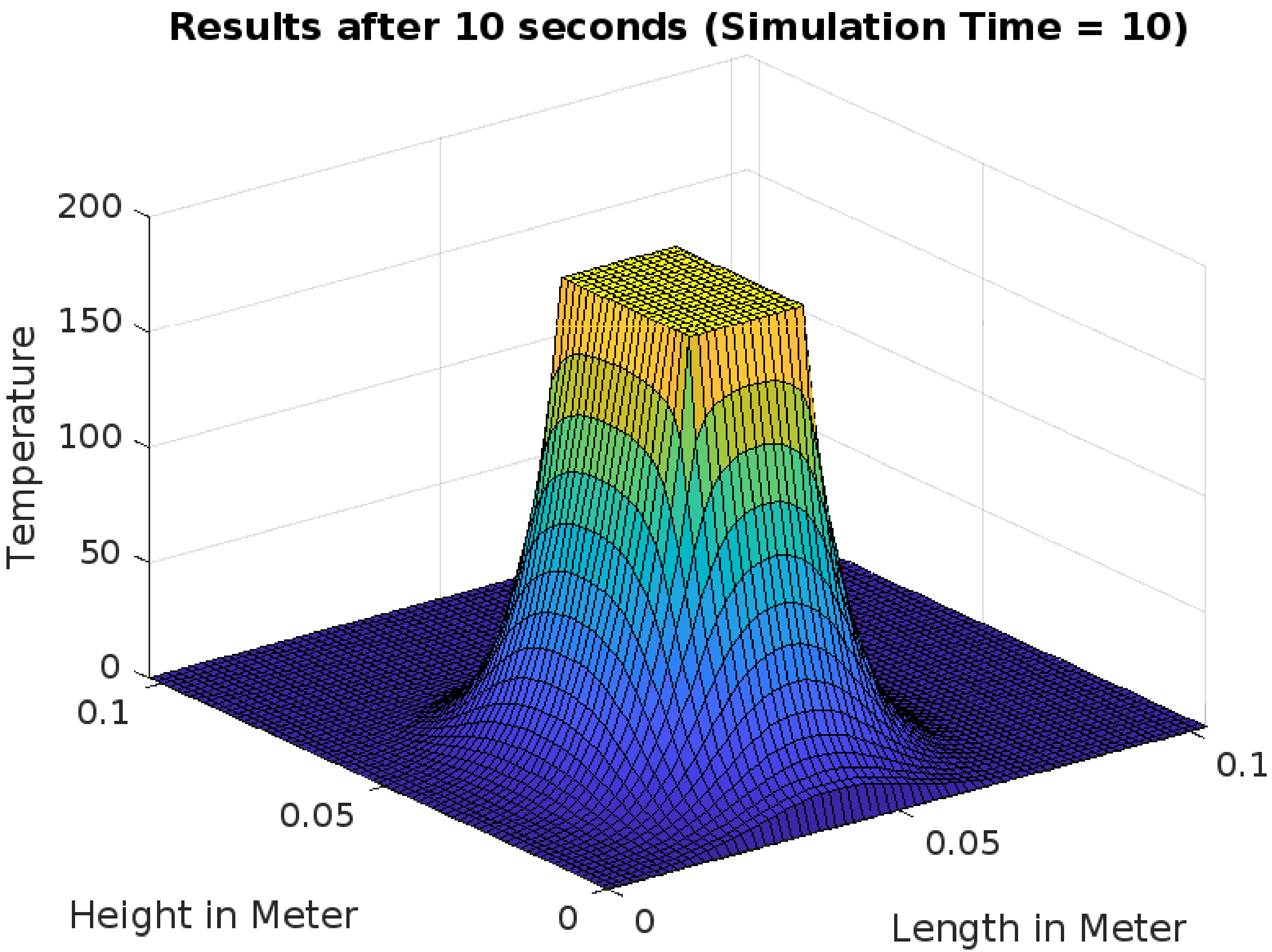


Figure 15: (a) and (b) are the simulated temperature contours from the plots of discretised contours (fig. 8) for 95% and 40% 'accuracy' respectively (Δt=0.05s and Δl = 0.001m and 0.008m)


Results after 10 seconds (Simulation Time = 10)
Temperature
200
150
100
50
0
0.1
0.05
0
Height in Meter
0
0.05
0.1
Length in Meter

Summary

- An intuitive guide for the selection of discretisation for transient parabolic models has been developed.
- TLM parameterisations have been developed and investigated for the representation of parabolic heat transfer.
- The transient properties of the TLM parameterisations has been investigated by simulations.
- Changes of choices of spatial discretisation produced geometrical variations that significantly influence temperature transient of simulation results.
- This research can be applied in Lager Soldering, Bio-Tech- Skin cancer, Breast cancer.

Any Questions?